

Preprint: September 26, 2007; December 21, 2007; January 8, 2008.
 Submitted to *Contr. Gen. Alg.* 18

THE TYPICAL COUNTABLE ALGEBRA

MARTIN GOLDSTERN

ABSTRACT. We argue that it makes sense to talk about “typical” properties of lattices, and then show that there is, up to isomorphism, a unique countable lattice L^* (the Fraïssé limit of the class of finite lattices) that has all “typical” properties. Among these properties are: L^* is simple and locally finite, every order preserving function can be interpolated by a lattice polynomial, and every finite lattice or countable locally finite lattice embeds into L^* .

The same arguments apply to other classes of algebras assuming they have a Fraïssé limit and satisfy the finite embeddability property.

1. WHAT IS A “TYPICAL” ALGEBRA?

1.1. Lattices. What is a “typical” property of a countably infinite lattice? It seems clear that the typical lattice will not be a chain, and will in fact have arbitrarily large finite antichains. Similarly, a “typical” lattice will contain arbitrarily long finite chains. It can also be argued that a typical lattice should not be distributive, since distributivity is a very special property for a lattice to have.

But why do we consider distributivity as a more special property than nondistributivity? One could argue that distributivity is a positive property, and that all “positive” properties are special; however, distributivity can also be seen as a negative property: neither N_5 nor M_3 embeds (see [Grätzer 1998, II.1, Theorem 1]).

We will define a property as “special” if only few countable lattices possess this property. Hence we need a means of measuring how “large” an infinite family of countable lattices is; as in [Goldstern 1997] we will use the topological notion of “first category” or “meagerness” for this purpose.

2000 *Mathematics Subject Classification.* Primary 08B25; Secondary 03C35, 08A55, 54E52.

Key words and phrases. Fraïssé limit, amalgamation, lattice, residual, ubiquitous, finite embeddability property, generic lattice.

Partially supported by the Austrian Science Foundation (FWF), grant P17627-N12.
 A preprint of this paper is available at arXiv.org.

1.2. General algebras. Our considerations will apply to a large class of algebras. Let \mathbb{L} be the set of countably infinite algebras of some variety. We want to describe the “typical” member of \mathbb{L} .

Definition 1. Let \mathbb{L} be a class of countable algebras. A finite \mathbb{L} -algebra is any finite subalgebra of an algebra in \mathbb{L} .

(Note that there may be finite algebras in the variety generated by \mathbb{L} which are not finite \mathbb{L} -algebras, e.g., if \mathbb{L} is the class of all countable Boolean algebras.)

On the set of all countable algebras in \mathbb{L} whose universe is equal to a fixed set $\mathbb{N} = \{0, 1, 2, \dots\}$ we will define a natural topology induced by a complete metric (Proposition 23). By Baire’s theorem, no nonempty open set is meager (=of first category). We say that “almost all countable algebras in \mathbb{L} have a certain property” if the set of all A in our space having this property is *residual* (i.e., has meager complement).

Theorem 2. Assume that the finite \mathbb{L} -algebras are a Fraïssé class (i.e., they have the amalgamation property (Definition 14) and the joint embedding property (Definition 18, see also Definition 19)). Assume further that \mathbb{L} has the finite embeddability property (every partial finite \mathbb{L} -algebra embeds into a finite \mathbb{L} -algebra, see Definition 10).

Let L^* be the Fraïssé limit (Theorem 22) of the finite \mathbb{L} -algebras.

Then almost all algebras in \mathbb{L} are isomorphic to L^* .

Corollary 3. Let L^* be as above. Let X be any property of algebras that is invariant under isomorphisms. Then there are two cases:

- (a) Almost all countable algebras in \mathbb{L} (including L^*) have property X . We call such a property “typical”.
- (b) Otherwise; then L^* does not have property X , so almost no countable algebra in \mathbb{L} has property X .

Hence L^* has all “typical” properties.

Remark 4. A variant of the above theorem for relational structures is well known under the keyword *ubiquitous*. [Bankston-Ruitenburg] credit the idea to a 1984 seminar talk by Peter Cameron. This notion and its variants have not only appeared in model theory but also in theoretical computer science, see [Droste-Kuske].

Algebras (and partial algebras) can of course be seen as relational structures (by replacing each n -ary function by the corresponding $(n + 1)$ -ary relation, see [Bankston-Ruitenburg, 7.7]).

Proofs are often easier if we deal with purely relational structures only: on an abstract level, the set of all countable structures for a relational language,

equipped with the natural (Tychonoff) topology, is a *compact* metric space, whereas the space of all countable algebras (see Proposition 23) is not compact. On a technical level, the Fraïssé limit (see Theorem 22) of a class of relational structures can be constructed as an increasing union of successive one-point extensions; in the algebraic setting this is not possible, unless we are ready to deal with partial algebras. The finite embedding property (see Definition 10) seems to be crucial for algebras, whereas it is irrelevant for relational structures, as relational structures are trivially locally finite.

The purpose of this note is to de-emphasise the role of compactness and to underline the role of the finite embeddability property in the algebraic context.

In Section 2 we explain (or recall) the definitions of a Fraïssé class and Fraïssé limit, and the finite embeddability property. Both notions are exemplified for the case of lattices. Rather than giving full proofs, we sketch proofs and mention the main ideas; the details can be found in textbooks ([Hodges] for model theoretic notions such as the Fraïssé limit, [Grätzer 1998] for lattice theory).

We prove the theorem in Section 3. Section 4 describes the example of lattices and bounded lattices.

2. BASIC NOTIONS

2.1. Ideals. Let U be any set, $I \subseteq \mathcal{P}(U)$ a proper ideal.

Definition 5. We say that “almost all” $u \in U$ have a property X iff the set

$$\{u \in U : u \text{ does not have property } X\}$$

is in the ideal I .

For definiteness we should say “ I -almost all” rather than just “almost all”. However, if the ideal I has a sufficiently natural definition, we may omit mentioning it. (The ideal we will use is the ideal of meager sets on a certain product space, see Proposition 23.)

Example 6. Let $U = [0, 1]$. Let I be the ideal of sets of Lebesgue measure zero.

Then, for example,

$$\text{“almost all } x \in [0, 1] \text{ are irrational”},$$

since the set of rational numbers has measure zero; a randomly picked number is unlikely to be rational.

2.2. Setup. Throughout this paper we fix a similarity type (also called signature) of algebras. For notational simplicity only we will assume that this type is (2,2), i.e., we will consider only algebras with two binary operations. We call these two operations \vee and \wedge . Later (see Section 2.5.1) we will restrict our attention to algebras whose universe is a subset of a fixed countable set \mathbb{N} .

We will also fix a class \mathbb{L} of countable algebras (typically the countable algebras of a variety).

2.3. Partial algebras and the finite embeddability property.

2.3.1. Weak and relative subalgebras. An algebra A is a set $\text{univ}(A)$ (called the universe of A) equipped with two binary operations \vee_A and \wedge_A . (We allow the universe of an algebra to be empty.)

A partial algebra is a set equipped with two (possibly) partial operations. The following definition is from [Grätzer 1979].

Definition 7. Let A be an algebra (or a partial algebra), and let $S \subseteq \text{univ}(A)$ be a subset of the universe of A . We write $A \upharpoonright S$ for the partial algebra that A induces on S , and we call $A \upharpoonright S$ the *relative subalgebra* of A determined by S . Thus, a relative subalgebra of A is any partial algebra B whose universe $\text{univ}(B)$ is a subset of the universe of A such that

$$a \wedge_A b = c \Leftrightarrow a \wedge_{A \upharpoonright S} b = c \quad (\text{and similarly for } \vee)$$

is satisfied for all $a, b, c \in \text{univ}(B)$.

A *weak subalgebra* of A is any partial algebra B whose universe is a subset of $\text{univ}(A)$ and whose operations, whenever defined, agree with the corresponding operations on A . E.g., whenever $a, b, c \in \text{univ}(B)$ and $a \wedge_B b = c$, then also $a \wedge_A b$ is defined and equal to c . (In contrast to the case of relative subalgebras, we allow here that $a \wedge_B b$ is undefined even when $a \wedge_A b$ is defined and an element of $\text{univ}(B)$.)

We write $B \leq_w A$ if B is a weak subalgebra of A , and $B \leq_r A$ if B is a relative subalgebra of A .

If B is a total algebra, then $B \leq_w A$ is equivalent to $B \leq_r A$; for total algebras A, B we abbreviate $B \leq_w A$ to $B \leq A$.

If $\delta : A \rightarrow B$ is a total isomorphism between partial algebras A and B , and C a partial algebra with $\text{univ}(B) \subseteq \text{univ}(C)$, then we write $\delta : A \hookrightarrow_w C$ iff $B \leq_w C$, and we say that δ embeds A weakly into C . The notation $A \hookrightarrow_w C$ means that there exists δ with $\delta : A \hookrightarrow_w C$. The relations $\delta : A \hookrightarrow_r C$ and $A \hookrightarrow_r C$ are defined similarly.

Remark 8. From the algebraic and model-theoretic point of view, the notion of a relative subalgebra seems to be more natural than the notion of a weak subalgebra. However, from the topological point of view the weak subalgebras

are more practical, since they correspond directly to a natural basis of the relevant topology (see Proposition 28). Lemma 9 shows that the distinction is irrelevant for our purposes.

2.3.2. Finite embeddability.

Lemma 9. *Let \mathbb{L} be a class of algebras which is closed under finite products and under isomorphic images. Then the following are equivalent:*

- (r,r) *Every finite relative subalgebra of some algebra in \mathbb{L} is a relative subalgebra of some finite algebra in \mathbb{L} .*
- (r,w) *Every finite relative subalgebra of some algebra in \mathbb{L} is a weak subalgebra of some finite algebra in \mathbb{L} .*
- (w,w) *Every finite weak subalgebra of some algebra in \mathbb{L} is a weak subalgebra of some finite algebra in \mathbb{L} .*

Proof. Trivially, the property (r,r) implies (r,w), as does the property (w,w).

It is also easy to see that (r,w) implies (w,w): Let $A \leq_w B \in \mathbb{L}$. Then we can find an algebra A' with the same universe as A which satisfies $A \leq_w A' \leq_r B$. From (r,w) we get a finite algebra $E \in \mathbb{L}$ with $A \leq_w A' \leq_r E$, which implies $A \leq_w E$.

We now prove that (w,w) implies (r,r). Let $A \leq_r B \in \mathbb{L}$. As \mathbb{L} is closed under isomorphism, it is enough to find a finite algebra E with $A \hookrightarrow_r E \in \mathbb{L}$. Let

$$I := \{a \wedge a', a \vee a' : a, a' \in \text{univ}(A)\} \setminus \text{univ}(A).$$

Note that $I \subseteq \text{univ}(B) \setminus \text{univ}(A)$. We may assume that $I \neq \emptyset$. [If $I = \emptyset$, then A is a total algebra, so $A \leq_w E$ trivially implies $A \leq_r E$ for any finite $E \in \mathbb{L}$.] For each $i \in I$ we define a partial algebra A_i with universe $\text{univ}(A) \cup \{i\}$ such that $A \leq_r A_i \leq_w B$; the only operations which are defined in A_i but not in A are those operations which map pairs of elements of A to i . For each $i \in I$ we can (using (w,w)) find a finite algebra $E_i \in \mathbb{L}$ such that $A_i \leq_w E_i$. Let $E := \prod_{i \in I} E_i$.

Let $\delta : \text{univ}(A) \rightarrow \text{univ}(E)$ be the natural embedding, defined by $\delta(a) = (a, \dots, a)$. Clearly $\delta : A \hookrightarrow_w E$, but we claim that even $\delta : A \hookrightarrow_r E$ holds.

So let $a, a' \in \text{univ}(A)$ and assume that (wlog) $a \wedge_A a'$ is undefined. We have to check that $\delta(a) \wedge_E \delta(a') \notin \delta(\text{univ}(A))$.

Let $i := a \wedge_B a'$. Then $i \in I$. The i -th component of $\delta(a) \wedge_E \delta(a') \in \text{univ}(E)$ is equal to $a \wedge_{E_i} a' = i$, hence $\delta(a) \wedge_E \delta(a')$ is not in the range of δ . \square

Definition 10. Let \mathbb{L} be a class of algebras which is closed under isomorphic images and under finite products. Following [Lindner-Evans] we say that \mathbb{L} has the *finite embeddability property* (f.e.p.) iff every finite *weak* subalgebra of some algebra in \mathbb{L} is a *weak* subalgebra of some finite algebra in \mathbb{L} . (Or

equivalently, iff every finite *relative* subalgebra of some algebra in \mathbb{L} is a *relative* subalgebra of some finite algebra in \mathbb{L} .)

Examples 11. Every locally finite variety has the f.e.p. We will see below that also the variety of lattices has the finite embeddability property.

I am grateful to Péter P. Pálfi for pointing out the following example:

Let G be the finitely presented group $\langle a, b \mid b^2a = ab^3 \rangle$. This group is non-Hopfian [Baumslag-Solitar], hence not residually finite [Robinson, 6.1.11]. So there is an element $g \in \text{univ}(G)$ which is contained in every normal subgroup of finite index. Let B be a finite partial subgroup of G which is generated by a and b and contains g , such that $b^2a = ab^3$ can be computed within B . If $\delta : B \rightarrow E$ is a homomorphism onto a finite group E , then E would be generated by $\delta(a)$ and $\delta(b)$ and satisfy $\delta(b)^2\delta(a) = \delta(a)\delta(b)^3$, so E would have to be a homomorphic image of G , and hence satisfy $\delta(g) = 1$.

This shows that the variety of groups does not have the f.e.p.

The following is an ad hoc example of an (admittedly artificial) variety where the failure of the f.e.p. is more obvious:

We have a binary operation $*$ and three unary operations p, q and F .

The equations of the variety say that on the range of F , $*$ is a bijection with inverses p and q :

$$p((Fx) * (Fy)) \approx Fx \approx q((Fy) * (Fx)), \quad p(Fz) * q(Fz) \approx Fz.$$

Then all finite algebras of the variety satisfy $Fx \approx Fy$.

So the 2-element partial algebra $\{a, b\}$ with $Fa = a$, $Fb = b$ (and $p, q, *$ undefined) is not a relative partial subalgebra of any finite algebra of the variety, although it is a relative partial subalgebra of some infinite algebra of the variety.

2.3.3. Funayama's theorem. [Funayama] (see also [Grätzer 1998, I.5, Theorem 20]) characterized the *partial lattices* (i.e., the partial algebras which are relative subalgebras of lattices) as those structures on which there is a well-behaved notion of “ideal” and shows that on each partial lattice P the map $x \mapsto (x)$ (which sends each x to the principal ideal generated by x) maps P isomorphically onto a relative sublattice of the lattice of ideals on P .

In particular (see [Grätzer 1998, I.5, Lemma 21]):

Proposition 12. Let (P, \leq) be a partial order, and define partial binary operations \vee and \wedge by

$$a \vee b := \sup(a, b) \quad a \wedge b := \inf(a, b) \quad (\text{whenever this is well-defined}).$$

Then the partial algebra (P, \wedge, \vee) is a partial lattice.

If P is finite then also the ideal lattice over P is finite, so as a corollary to Funayama's theorem we get:

Proposition 13. The variety of lattices has the f.e.p. That is: Whenever $P = (\text{univ}(P), \vee_P, \wedge_P)$ is a finite relative subalgebra of a lattice, then P is a relative subalgebra of a *finite* lattice.

2.4. Fraïssé classes.

2.4.1. *Amalgamation and joint embedding.* We will need the following notation: If A, B, B' are algebras with $A \leq B$ and $A \leq B'$, we write $B \simeq_A B'$ (" B is isomorphic to B' over A ") iff there exists an isomorphism $g : B \rightarrow B'$ which is the identity map on A .

Definition 14. Let A, B_1, B_2, D be algebras and let $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$ be 1-1 homomorphisms.

We say that D is an amalgamation of B_1 and B_2 over A if there are 1-1 homomorphisms $g_1 : B_1 \rightarrow D$, $g_2 : B_2 \rightarrow D$ such that $g_1 \circ f_1 = g_2 \circ f_2$. (See Figure 1.)

(More precisely we say that (D, g_1, g_2) is an amalgamation of B_1 and B_2 over A , or over f_1, f_2 . Intuitively it means that B_1 and B_2 are glued together, identifying $f_1(A)$ with $f_2(A)$.)

The following fact was noted in [Jónsson]:

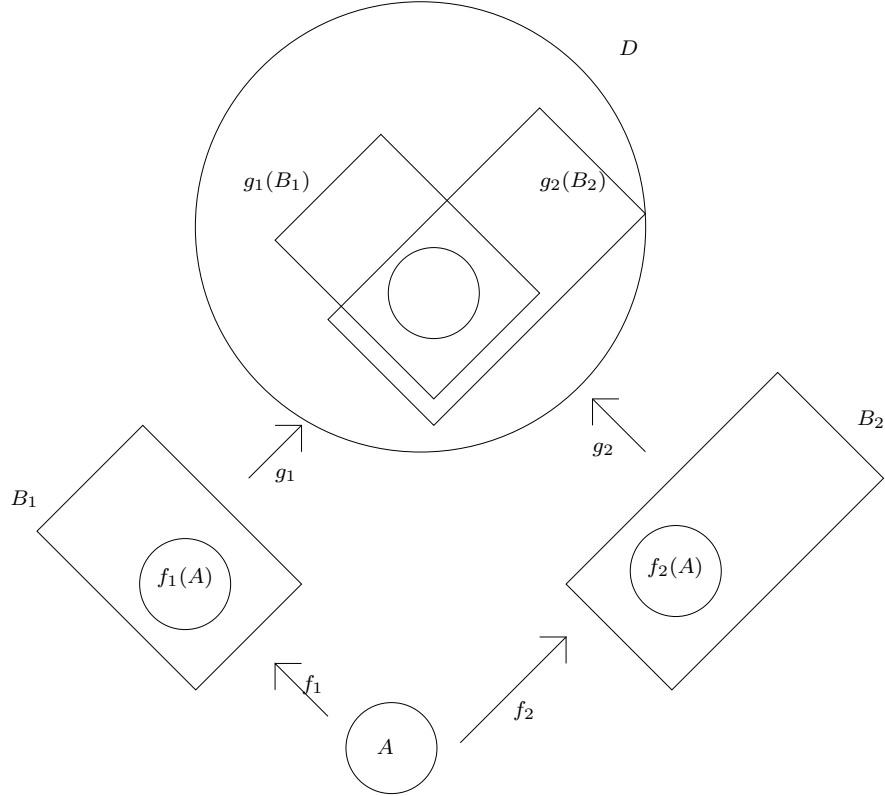
Proposition 15. The family of all lattices has the *amalgamation property*, that is:

Whenever $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$ are 1-1 homomorphisms between lattices, then there exists a lattice D which amalgamates B_1 and B_2 over A .

This can be proved using Funayama's characterization: We may assume that f_1 and f_2 are inclusion maps, and that $A = B_1 \cap B_2$. The set $\text{univ}(B_1) \cup \text{univ}(B_2)$ is naturally partially ordered by the transitive closure of the union of the orders \leq_{B_1} and \leq_{B_2} ; by Proposition 12, the partial operations $\sup(x, y)$ and $\inf(x, y)$ make this poset into a partial lattice $B_1 \cup B_2$, so that the amalgamation of B_1 and B_2 over A can be taken to be the set of ideals of $B_1 \cup B_2$. (See [Grätzer 1998, Section V.4] for a detailed version of this proof.)

Combining this fact with the finite embeddability property we easily see:

Proposition 16. The amalgamation of two finite lattices is (or: can be chosen to be) finite. In other words: The class of finite lattices has the amalgamation property.

FIGURE 1. Amalgamation of B_1 and B_2 over A

If f_1 and f_2 are the identity embeddings, then we can also identify B_1 with $g_1(f_1(B_1)) = g_1(B_1)$ and get the following convenient reformulation:

Proposition 17. Let \mathbb{L} be a class of algebras with the amalgamation property which is closed under isomorphism. Let $A \leq B_1$, $A \leq B_2$, and A, B_1, B_2 be algebras in \mathbb{L} .

Then we can find algebras B'_2 and D in \mathbb{L} with (see Figure 2):

$$B'_2 \leq D, \quad A \leq B_1 \leq D, \quad \text{and} \quad B_2 \simeq_A B'_2.$$

Definition 18. We say that a family \mathbb{L} of structures has the joint embedding property if for any structures $A, B \in \mathbb{L}$ there is a structure $C \in \mathbb{L}$ which contains isomorphic copies of both A and B . If \mathbb{L} contains the empty structure (or more generally: has an initial element), then the amalgamation property of \mathbb{L} implies the joint embedding property for \mathbb{L} .

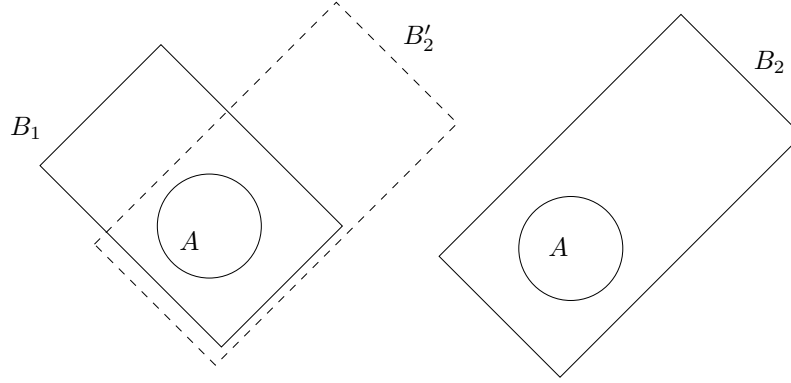


FIGURE 2. Amalgamation with inclusion

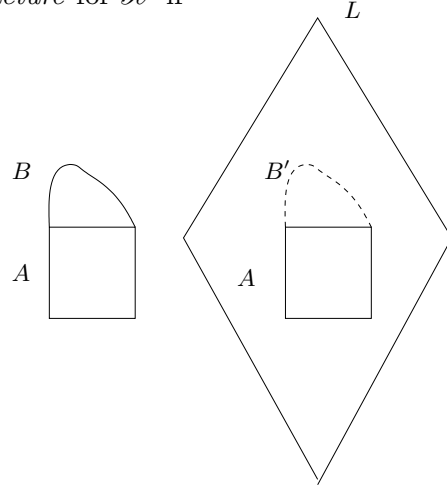
Definition 19. We call a family \mathbb{L} of structures a *Fraïssé class* if \mathbb{L} is closed under substructures, and has the amalgamation property and the joint embedding property.

Proposition 20. The family of lattices and the family of finite lattices are Fraïssé classes.

2.4.2. Ultrahomogeneity and the Fraïssé limit.

Definition 21. Let \mathcal{K} be a class of finite algebras, and let L be a countable algebra. We say that L is a *Fraïssé structure* for \mathcal{K} if

- (U1) For all $A, B \in \mathcal{K}$:
 If $A \leq B$ and $A \leq L$,
 then there is $B' \in \mathcal{K}$ with
 $A \leq B' \leq L$ and $B \simeq_A B'$.
 (We call this property “ultra-
 homogeneity”; note that this
 word is often used for the
 following related notion, see
 Theorem 22(b): Every finite
 partial isomorphism between
 [partial] substructures of L
 extends to an automorphism
 of L .)
- (U2) L contains isomorphic copies of all elements of \mathcal{K} .
 (This is sometimes written as “ L is universal”. If \mathcal{K} contains the
 empty algebra or more generally has an initial element, then this prop-
 erty follows from U1.)
- (U3) All finite substructures of L are in \mathcal{K} .



If \mathcal{K} is a Fraïssé class of finite algebras, then we can inductively build a countable Fraïssé structure for \mathcal{K} . We construct an increasing sequence $(C_n : n = 1, 2, \dots)$ of finite algebras such that

- for all $A, B \in \mathcal{K}$ and all n we have: if $A \leq B$ and $A \leq C_n$, then there is $n' > n$ and $A \leq B' \leq C_{n'}$ with $B \simeq_A B'$;
- for all $B \in \mathcal{K}$ there is n' and $B' \leq C_{n'}$ with $B \simeq B'$.

The result of such a construction is called the *Fraïssé limit* of \mathcal{K} . It is unique up to isomorphism:

Theorem 22 ([Fraïssé], see also [Hodges, Chapter 6.1]). *Let \mathcal{K} be a Fraïssé class of finite algebras. Then:*

- (a) *There is a unique locally finite countable structure L^* which is a Fraïssé structure for \mathcal{K} .*
- (b) *Moreover, if L_1 and L_2 are both locally finite Fraïssé structures for \mathcal{K} , then every partial isomorphism between finite subalgebras of L_1 and L_2 can be extended to an isomorphism from L_1 onto L_2 .*
- (c) *Every countable locally finite algebra is isomorphic to a subalgebra of L^* , assuming that all its finite subalgebras are in \mathcal{K} .*

2.5. Topology. Let X be a complete metric space. A subset $M \subseteq X$ is called “nowhere dense” if there is no open set contained in the topological closure of M , and M is called meager (or: “of first category”), if M can be covered by countably many nowhere dense (or: nowhere dense closed) sets.

Clearly, the family of meager sets forms an ideal. Since X is a complete metric space, Baire’s theorem tells us that this ideal is proper: X itself is not meager. (In fact, no nonempty open set is meager.)

2.5.1. A metric space of algebras. Recall that for notational simplicity we only consider algebras with 2 binary operations.

We now consider all algebras (with two binary operations) whose underlying set is the set \mathbb{N} of natural numbers. A binary operation on \mathbb{N} is just a map from $\mathbb{N} \times \mathbb{N}$ into \mathbb{N} , i.e., an element of $\mathbb{N}^{\mathbb{N}^2}$. Identifying each algebra $A = (\mathbb{N}, \vee_A, \wedge_A)$ with the pair $(\vee_A, \wedge_A) \in \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2}$, we see that our set of algebras is really the set $\mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2}$, which we abbreviate as \mathcal{N} .

The space \mathcal{N} in the next proposition is sometimes called “Baire space”; it is homeomorphic to the set of irrational real numbers.

Proposition 23. Using the discrete topology on \mathbb{N} , and the Tychonoff topology (product topology) on $\mathbb{N}^{\mathbb{N}^2}$ and $\mathcal{N} = \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2}$, the space \mathcal{N} is a perfect Polish space (i.e., separable and completely metrizable).

An example of a complete metric on \mathcal{N} is given by $d((f, g), (f', g')) = 2^{-n}$, where n is minimal such that there exist $i, j \leq n$ with $f(i, j) \neq f'(i, j)$ or $g(i, j) \neq g'(i, j)$.

Proposition 24. Let \mathbb{L} be a variety. Then $\mathbb{L} \cap \mathcal{N}$ is a closed subset of \mathcal{N} , hence also a Polish space.

Definition 25.

- $\mathcal{N} := \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2}$ is the set of all algebras on the fixed countable set \mathbb{N} .
- $\mathbb{L}_{\upharpoonright \mathcal{N}} := \mathbb{L} \cap \mathcal{N}$.
- We write $\mathbb{L}_{\upharpoonright \mathcal{N}, \text{locfin}}$ for the locally finite algebras in $\mathbb{L}_{\upharpoonright \mathcal{N}}$.
- We write $\mathbb{L}_{\upharpoonright \mathcal{N}, \text{fin}}$ for the class of all finite algebras whose universe is a subset of \mathbb{N} .
- We write $\mathbb{L}_{\upharpoonright \mathcal{N}, w}$ for the class of all finite partial algebras which are weak subalgebras of an algebra in $\mathbb{L}_{\upharpoonright \mathcal{N}}$.

Definition 26. “Almost all countable algebras in \mathbb{L} have property X ” will mean:

The set $\{L \in \mathbb{L}_{\upharpoonright \mathcal{N}} : L \text{ does not have property } X\}$ is meager (in the complete metric space $\mathbb{L}_{\upharpoonright \mathcal{N}}$).

Definition 27. For any finite partial algebra $P \in \mathbb{L}_{\upharpoonright \mathcal{N}, w}$ we let

$$[P]_w := \{L \in \mathbb{L}_{\upharpoonright \mathcal{N}} : P \leq_w L\}.$$

If P is a total algebra, we write $[P]$ instead of $[P]_w$.

Note that for each $a, b, c \in \text{univ}(P)$ the sets $\{L \in \mathbb{L}_{\upharpoonright \mathcal{N}} : a \wedge_L b = c\}$ and $\{L \in \mathbb{L}_{\upharpoonright \mathcal{N}} : a \vee_L b = c\}$ are clopen, by the definition of the product topology in $\mathbb{L}_{\upharpoonright \mathcal{N}}$; hence $[P]_w$ is clopen, as a finite intersection of clopen sets. A closer inspection of the open sets in the product topology yields the following proposition:

Proposition 28. The family $\{[P]_w : P \in \mathbb{L}_{\upharpoonright \mathcal{N}, w}\}$ is a clopen basis for the topology on $\mathbb{L}_{\upharpoonright \mathcal{N}}$. In other words: For all $L \in \mathbb{L}_{\upharpoonright \mathcal{N}}$, every open neighborhood of L contains a neighborhood of the form $[P]_w$, for some $P \in \mathbb{L}_{\upharpoonright \mathcal{N}, w}$, $P \leq_w L$. (Here, P is a finite *partial* algebra.)

For locally finite L we can ignore partial algebras altogether: If $L \in \mathbb{L}_{\upharpoonright \mathcal{N}, \text{locfin}}$, then every open neighborhood of L contains a neighborhood of the form $[A]$, for some $A \in \mathbb{L}_{\upharpoonright \mathcal{N}, \text{fin}}$, $A \leq L$. (Here, A is a finite *total* algebra.)

3. THE TYPICAL COUNTABLE ALGEBRA

Throughout this section the class \mathbb{L} will be the class of countable algebras of some fixed variety.

We write $\mathbb{L}_{\upharpoonright \mathcal{N}}$ for the set of all algebras in \mathbb{L} whose carrier set is the set \mathbb{N} of natural numbers. At various points we will assume that \mathbb{L} has several properties, in particular: the finite embeddability property, the amalgamation property and the joint embedding property. If we take \mathbb{L} to be the class of countable lattices, then \mathbb{L} has all these properties.

3.1. Two residual subsets of $\mathbb{L}_{\downarrow \mathcal{N}}$. Our first lemma shows that the f.e.p. allows us to restrict our attention to locally finite algebras; thus the neighborhoods in our topological space are generated already by the finite total algebras and we may ignore partial algebras.

Lemma 29. *Assume that \mathbb{L} has the finite embeddability property. Then almost all algebras $L \in \mathbb{L}_{\downarrow \mathcal{N}}$ are locally finite.*

In fact, the set $\mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}}$ of locally finite algebras is the intersection of countably many dense open sets of $\mathbb{L}_{\downarrow \mathcal{N}}$ (and hence, with the subspace topology, also a Polish space).

Proof. We have $\mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}} = \bigcap_{k \in \mathbb{N}} \bigcup_{\substack{\{0, \dots, k\} \subseteq \text{univ}(B) \\ B \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}}} [B]$.

We claim that each set

$$\mathcal{L}_k := \{L \in \mathbb{L}_{\downarrow \mathcal{N}} : \exists B \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}, B \leq L, \{0, \dots, k\} \subseteq \text{univ}(B)\}$$

is open dense. It is clear that \mathcal{L}_k is open.

Now fix any nonempty open subset of $\mathbb{L}_{\downarrow \mathcal{N}}$. By Proposition 28 we may assume that this open set is of the form $[P]_w$, where $P \in \mathbb{L}_{\downarrow \mathcal{N}, w}$ is a partial algebra. Find a partial algebra $P' \in \mathbb{L}_{\downarrow \mathcal{N}, w}$, $P \leq_w P'$ such that $\text{univ}(P')$ contains the set $\{0, \dots, k\}$. By the finite embeddability property, there is a total algebra $B \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}$, $P' \leq B$. So $[P]_w \supseteq [P']_w \supseteq [B]$. $[B]$ is nonempty, so \mathcal{L}_k meets $[P]_w$. So \mathcal{L}_k is dense. \square

Proposition 30. Assume that the class \mathbb{L} is a Fraïssé class with the finite embeddability property. Then the set

$$\mathcal{U} := \{L \in \mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}} : L \text{ is Fraïssé for } \mathbb{L}\}$$

is residual in $\mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}}$, hence also in $\mathbb{L}_{\downarrow \mathcal{N}}$.

Proof. We can write \mathcal{U} as $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, where for $\ell = 1, 2$ the set \mathcal{U}_ℓ is the set of all $L \in \mathbb{L}_{\downarrow \mathcal{N}}$ satisfying property \mathcal{U}_ℓ in Definition 21. We will show that \mathcal{U}_1 is residual, leaving the case of \mathcal{U}_2 to the reader.

We have

$$\begin{aligned} \mathcal{U}_1 &= \bigcap_{A \leq B \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}} \{L : \text{if } A \leq L \text{ then } \exists B' \leq L : B \simeq_A B'\} = \\ &= \bigcap_{A \leq B \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}} (\mathcal{U}_{\neg A} \cup \mathcal{U}_{A, B}), \end{aligned}$$

where $\mathcal{U}_{\neg A} = \mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}} \setminus [A]$ is the set of all $L \in \mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}}$ with $A \not\leq L$, and

$$\mathcal{U}_{A, B} := \{L \in \mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}} : A \leq L, \text{ and } \exists B' \leq L : B \simeq_A B'\}.$$

The sets $\mathcal{U}_{\neg A}$ and $\mathcal{U}_{A,B}$ are open ($\mathcal{U}_{A,B}$ is the union of sets of the form $[B']$, $B' \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}$).

We now check that each set $\mathcal{U}_{\neg A} \cup \mathcal{U}_{A,B}$ is dense.

So let $[C]$ be a basic open set in $\mathbb{L}_{\downarrow \mathcal{N}, \text{locfin}}$, $C \in \mathbb{L}_{\downarrow \mathcal{N}, \text{fin}}$. We may assume that $\text{univ}(A) \subseteq \text{univ}(C)$. If $A \not\leq C$, then $[C] \subseteq \mathcal{U}_{\neg A}$, so we are done.

So assume $A \leq C$. As also $A \leq B$, we can find a finite algebra D and a subalgebra B' such that $B \simeq_A B'$ (using Proposition 17). Wlog $\text{univ}(D) \subseteq \mathbb{N}$. We have $[D] \subseteq [C]$, and $[D] \subseteq \mathcal{U}_{A,B}$. \square

Corollary 31. *Assume that \mathbb{L} is a Fraïssé class with the joint embedding property. Let L^* be the Fraïssé limit of the finite algebras in \mathbb{L} . Then almost all countable algebras in $\mathbb{L}_{\downarrow \mathcal{N}}$ are Fraïssé structures for the finite structures in \mathbb{L} , and hence are isomorphic to L^* .*

In particular, taking \mathbb{L} = the class of all countable lattices, we get: There is a (“typical” or “generic”) countable lattice L^* such that almost all countable lattices are isomorphic to L^* .

4. EXAMPLES

Assume that \mathbb{L} has the finite embeddability property, the amalgamation property and the joint embedding property. Let L^* be the Fraïssé limit of the finite algebras in \mathbb{L} .

We have already seen that L^* is locally finite and contains copies of all finite \mathbb{L} -algebras, and (by Theorem 22(c)) even all locally finite countable \mathbb{L} -algebras. The finite embeddability property implies that L^* satisfies no law that does not hold in all \mathbb{L} -algebras; in other words, L^* generates \mathbb{L} .

Since every finite automorphism between subalgebras (even partial subalgebras) extends to an automorphism of L^* , L^* has a very rich automorphism group.

The typical countable Boolean algebra is the (unique) atomless countable Boolean algebra; the typical distributive lattice and its automorphism group have been investigated in [Droste-Macpherson].

We now consider two special cases which are not locally finite: lattices and bounded lattices.

4.1. Lattices. Let L^* be the typical countable lattice.

As remarked in the introduction, L^* is certainly not distributive (it contains N_5), and in fact satisfies no law that is not implied by laws defining the variety of lattices. Since L^* contains all countable locally finite lattices, L^* contains, for example, a chain isomorphic to the rational numbers, and also an infinite antichain.

Note that L^* is very different from the random order R , i.e., the Fraïssé limit of the class of finite partial orders: in R there are *no* elements $x \neq y$ which have a smallest upper bound.

Proposition 32. For every monotone function $f : (L^*)^n \rightarrow L^*$ and every finite sublattice $A \leq L^*$ there is a lattice polynomial $p(x_1, \dots, x_n)$ which induces the function f on A . (We say that L^* has the *monotone interpolation property*.)

This implies that L^* is simple (i.e., has no nontrivial congruence relations).

Proof. By [Goldstern 1996], we can find a finite lattice B , $A \leq B$ and a polynomial p with coefficients in B which interpolates f on $\text{univ}(A)$, i.e., $p(\vec{a}) = f(\vec{a})$ for all $\vec{a} \in A^n$. We can use the ultrahomogeneity of L^* to find $B' \leq L^*$, $B \simeq_A B'$. The isomorphism between B and B' translates p to a polynomial p' with coefficients in B' which still interpolates f on A .

The fact that the monotone interpolation property implies that there are no nontrivial congruence relations is well known: for any nontrivial congruence relation \sim we can find $a_1 < a_2$ and $b_1 < b_2$ such that $a_1 \sim a_2$, $b_1 \not\sim b_2$. There is a monotone total function mapping a_i to b_i ; as such a function does not respect \sim , such a function cannot be a polynomial. \square

Proposition 33. Any two nontrivial intervals in the typical lattice are isomorphic to each other. (In fact, the isomorphism can be taken to be the restriction of an automorphism of L^* .)

Proof. By Theorem 22(b). \square

4.2. $\{0, 1\}$ -lattices. We now consider a language where we have two constant symbols $0, 1$ in addition to the two operations \wedge and \vee . We will consider the variety of $\{0, 1\}$ -lattices (i.e., lattices in which 0 and 1 are the greatest and smallest element).

Note that the 1-element $\{0, 1\}$ -lattice does not embed into any other $\{0, 1\}$ -lattice, so we will have to ignore it in our considerations. **From now on, $\{0, 1\}$ -lattice will mean: $\{0, 1\}$ -lattice with $0 \neq 1$.**

The same construction as in Proposition 15 (after the necessary change of notation, taking into account the new constants) shows the following:

Proposition 34. Whenever A is a $\{0, 1\}$ -lattice, and $f_1 : A \rightarrow B_1$, $f_2 : A \rightarrow B_2$ are $\{0, 1\}$ -homomorphisms, then there exists a $\{0, 1\}$ -lattice D and $\{0, 1\}$ -homomorphisms $g_i : B_i \rightarrow D$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Moreover, if B_1 and B_2 are finite, then D can be chosen to be finite.

In other words: the class of $\{0, 1\}$ -lattices, as well as the class of finite $\{0, 1\}$ -lattices, has the amalgamation property.

Proposition 35. The class of $\{0, 1\}$ -lattices has the finite embeddability property.

Proof. Let A be a partial finite $\{0, 1\}$ -lattice. Wlog we may assume that 0 and 1 are defined in A . The natural embedding of A into the lattice of *nonempty* ideals preserves 0 and 1 (since 0 is mapped to the smallest nonempty ideal $\{0\}$). \square

It is now easy to see that the $\{0, 1\}$ -lattices are a Fraïssé class with the f.e.p. Hence there is a “typical” countable $\{0, 1\}$ -lattice K^* . Note that 1 is typically not join-irreducible (i.e., there are lattices, even finite ones, where $1 = x \vee y$ for some $x, y < 1$), so 1 is also not join-irreducible in K^* .

Let L^* be the typical lattice, and let $a, b \in L^*$, $a < b$. Consider the interval $[a, b]$ as a $\{0, 1\}$ -lattice with $a = 0$, $b = 1$.

The ultrahomogeneity/universality of L^* (with respect to the class of finite lattices) easily implies that $[a, b]$ is ultrahomogeneous/universal (with respect to the class of finite $\{0, 1\}$ -lattices). Hence we get:

Proposition 36. Every nontrivial interval in the typical lattice L^* is isomorphic to every nontrivial interval in the typical $\{0, 1\}$ -lattice K^* , in particular to K^* itself.

Using [Goldstern 1998] instead of [Goldstern 1996] we can also show:

Proposition 37. For every monotone function $f : (K^*)^n \rightarrow K^*$ and every finite $\{0, 1\}$ -sublattice $A \leq K^*$ there is a lattice polynomial $p(x_1, \dots, x_n)$ which induces the function f on A .

This again implies that K^* is simple.

5. ACKNOWLEDGEMENTS

I am grateful to Manfred Droste for pointing out several papers on ubiquitous structures, to Michael Pinsker for a conversation that led to Lemma 9, to Günther Eigenthaler for his careful proofreading, and to both referees for their constructive criticism.

REFERENCES

- [Bankston-Ruitenburg] Paul Bankston and Wim Ruitenburg. Notions of relative ubiquity for invariant sets of relational structures. *J. Symbolic Logic*, 55(3):948–986, 1990.
- [Baumslag-Solitar] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962.

- [Droste-Kuske] Manfred Droste and Dietrich Kuske. Almost every domain is universal. *Electronic Notes in Theoretical Computer Science*, 173:103–119, 2007.
- [Droste-Macpherson] Manfred Droste and Dugald Macpherson. The automorphism group of the universal distributive lattice. *Algebra Universalis*, 43(4):295–306, 2000.
- [Fraïssé] Roland Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000. With an appendix by Norbert Sauer.
- [Funayama] Nenosuke Funayama. Notes on lattice theory. IV. On partial (semi-) lattices. *Bull. Yamagata Univ. (Nat. Sci.)*, 2:171–184, 1953.
- [Goldstern 1996] Martin Goldstern. Interpolation of monotone functions in lattices. *Algebra Universalis*, 36:108–121, 1996.
- [Goldstern 1997] Martin Goldstern. Most algebras have the interpolation property. *Algebra Universalis*, 38:97–114, 1997.
- [Goldstern 1998] Martin Goldstern. Interpolation of monotone functions in $\{0, 1\}$ -lattices. In *Contributions to General Algebra 10*. Heyn Verlag, 1998.
- [Grätzer 1979] George Grätzer. *Universal algebra*. Springer-Verlag, 1979.
- [Grätzer 1998] George Grätzer. *General lattice theory. 2nd ed.* Birkhäuser, Basel, 1998.
- [Hodges] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, 1997.
- [Jónsson] Bjarni Jónsson. Universal relational systems. *Math. Scand.*, 4:193–208, 1956.
- [Lindner-Evans] Charles C. Lindner and Trevor Evans. *Finite embedding theorems for partial designs and algebras*. Les Presses de l'Université de Montréal, Montreal, Que., 1977. Séminaire de Mathématiques Supérieures, No. 56 (Été 1971).
- [Robinson] Derek J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

ALGEBRA/DMG, TU WIEN, WIEDNER HAUPTSTR 8-10/104, 1040 WIEN, AUSTRIA

E-mail address: goldstern@tuwien.ac.at

URL: <http://www.tuwien.ac.at/goldstern/>